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On The Number of Representations of a Positive Integer By Certain Binary Quadratic Forms Belonging To Multi-Class Genera

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Abstract
It is well known how to find the formulae for the number of representations of positive integers by the positive binary quadratic forms which belong to one-class genera. In this paper we obtain the formulae for the number of representations by certain binary forms with discriminants –80 and –128 belonging to the genera having two classes**.**
Key words – positive integer, binary form, genera, number of representations.

I. Introduction

Let $r(n;f)$ denote a number of representations of a positive integer n by a positive definite quadratic form f with a number of variables *s*. It is well known that, for case $s>4$, $r(n;f)$ can be represented as

$$r(n;f)=\rho(n;f)+\nu(n;f),$$

where $\rho(n;f)$ is a "singular series" and $\nu(n;f)$ is a Fourier coefficient of cusp form. This can be expressed in terms of the theory of modular forms by stating that

$$\vartheta(\tau;f)=E(\tau;f)+X(\tau), \qquad (1)$$

$$\vartheta(\tau;f)=1+\sum_{n=1}^{\infty} r(n;f)Q^n,$$

where $\tau\in H=\{\tau:\operatorname{Im}\tau>0\}$, $Q=e^{2\pi i\tau}$, $X(\tau)$ is a cusp form and

$$E(\tau,f)=1+\sum_{n=1}^{\infty}\rho(n;f)Q^n$$

is the Eisenstein series corresponding to f.
Siegel [1] proved that if the number of variables of a quadratic form f is $s>4$, then

$$E(\tau;f)=F(\tau;f), \qquad (2)$$

where $F(\tau;f)$ denote a theta-series of a genus containing a primitive quadratic form f (both positive-definite and indefinite).
From formula (2) follows the well-known Siegel's theorem [1]: the sum of the singular series corresponding to the quadratic form f is equal to the average number of representations of a natural number by a genus that contains the form f.
Later Ramanathan [2] proved that for any primitive integral quadratic form with $s\geq 3$ variables (except for zero forms with variables $s=3$ and zero forms with variables $s=4$ whose discriminant is a perfect square), there is a function

$$E(\tau,z;f)=1+\frac{e^{\frac{(2m-s)\pi i}{4}}}{|d|^{\frac{1}{2}}}\times$$

$$\times\sum_{q=1}^{\infty}\sum_{\substack{H=-\infty\\(H,q)=1}}^{\infty}\frac{S(fH;q)}{q^{\frac{s}{2}}(q\tau-H)^{\frac{m}{2}}(q\bar{\tau}-H)^{\frac{s-m}{2}}|q\tau-H|^{z}},$$

which he called the Eisenstein-Siegel series and which is regular for any fixed τ when $\operatorname{Im}\tau>0$ and $\operatorname{Re} z>2-\frac{s}{2}$, analytically extendable in a neighborhood of $z=0$, and that

$$F(\tau;f)=E(\tau,z;f)\big|_{z=0} \qquad (3)$$

Here *m* and *d* are respectively the inertia index and discriminant of *f*, $S(fH;q)$ is the Gaussian sum. For $s>4$ the function $E(\tau,z;f)\big|_{z=0}$ coincides with the function $E(\tau,f)$ and the formula (3) with Siegel's formula (2).
In [3] we proved that the function $E(\tau,z;f)$ is analytically extendable in a neighborhood of $z=0$ also in the case where f is any nonzero integral binary quadratic form and that $F(\tau;f)=\frac{1}{2}E(\tau,z;f)\big|_{z=0}$ further, having defined the Eisenstein's series $E(\tau,f)$ by the formulas

$E(\tau;f)=\frac{1}{2}E(\tau,z;f)\big|_{z=0}$ for $s=2$ and

$E(\tau;f)=E(\tau,z;f)\big|_{z=0}$ for $s>2$, we have

$E(\tau,f)=1+\frac{1}{2}\sum_{n=12}^{\infty}\rho(n;f)Q^n$ for $s=2$ and

$E(\tau, f) = 1 + \sum_{n=1}^{\infty} \rho(n; f) Q^n$ for $s > 2$. Moreover, convenient formulas are obtained for calculating of values of the function $\rho(n; f)$ in the case where *f* is any positive integral form of variables $s \geq 2$ [4], [5].

Thus, if the genus of the quadratic form *f* contains one class, then according to Siegel's theorem, $r(n; f) = \; = \rho(n; f)$ for $s \geq 3$, $r(n; f) = \frac{1}{2} \rho(n; f)$ for $s = 2$ and in that case the problem for obtaining "exact" formulas for $r(n; f)$ is solved completely.

Some papers are devoted to the case of multi-class genera. For example Van der Blij [6], Lomadze [7], Vepkhvadze [5] have obtained formulae for $r(n; f)$ for certain special binary forms, which belong to multi-class genera. In most cases their formulae for $r(n; f)$ depend upon the coefficients in the expansion of certain products of theta functions.

In this paper we obtain formulae for the number of representations by the binary quadratic forms $3x_1^2 + 2x_1x_2 + 7x_2^2$, $3x_1^2 - 2x_1x_2 + 7x_2^2$, $3x_1^2 + 2x_1x_2 + 11x_2^2$ and $3x_1^2 - 2x_1x_2 + 11x_2^2$ belonging to the two-class genera.

II. Preliminaries

Below in this paper *n*, κ, ν, ω are positive integers; *l*, æ, α, β are non-negative integers; *u* is an odd positive integer, *m*, *h* are integers, *p* is a prime number; $\left(\frac{h}{u}\right)$ is a Legendre-Jacobi symbol; $r(n; f)$ denote the number of representations of *a* positive integer *n* by the positive binary quadratic form *f* with determinant Δ.

Lemma 1 ([4], Lemma 14). Let

$$L(\kappa; m) = \sum_{u=1}^{\infty} \left(\frac{m}{u}\right) \frac{1}{u^{\kappa}},$$

ω is *a* square-free integer. Then

$$L(1, -\omega) = \frac{\pi}{4} \quad \text{if} \quad \omega = 1;$$

$$= \frac{\pi}{\omega^{1/2}} \sum_{1 \leq h \leq \frac{\omega}{4}} \left(\frac{h}{\omega}\right) \quad \text{if} \quad \omega \equiv 1 \pmod 4,$$

$\omega > 1$,

$$= \frac{\pi}{2\omega^{1/2}} \sum_{1 \leq h \leq \frac{\omega}{2}} \left(\frac{h}{\omega}\right) \quad \text{if} \quad \omega \equiv 3 \pmod 4,$$

$$= \frac{\pi}{2^{3/2}} \quad \text{if} \quad \omega = 2,$$

$$= \frac{\pi}{\omega^{1/2}} \left\{ \sum_{1 \leq h \leq \frac{\omega}{16}} \left(\frac{h}{\frac{1}{2}\omega}\right) - \sum_{\frac{3\omega}{16} < h \leq \frac{\omega}{4}} \left(\frac{h}{\frac{1}{2}\omega}\right) \right\}$$

if $\omega \equiv 2 \pmod 8$, $\omega > 2$,

$$= \frac{\pi}{\omega^{1/2}} \sum_{\frac{\omega}{16} < h \leq \frac{3\omega}{16}} \left(\frac{h}{\frac{1}{2}\omega}\right) \quad \text{if} \quad \omega \equiv 6 \pmod 8.$$

Lemma 2 ([4], Theorem 2). Let $f = ax^2 + 2bxy + cy^2$ is a positive binary quadratic form with determinant $\Delta = ac - b^2$, $(a, 2b, c) = 1$, $(a, 2\Delta) = 1$, $\Delta = r^2\omega$, $\Delta = 2^{\text{æ}}\Delta_1 (2 \nmid \Delta_1)$, $p^l \parallel \Delta$, $p^{\beta} \parallel n$. $\rho(n; f)$ is a "sum of a generalized singular series" corresponding to *f*, $n = 2^{\alpha} m$, $p > 2$, $p^l \parallel \Delta$, $p^{\beta} \parallel n$, $u = \prod_{\substack{p \mid n \\ p \nmid 2\Delta}} p^{\beta}$. Then

$$\rho(n; f) = \frac{\pi \chi_2 \prod_{p/\Delta,\, p>2} \chi_p \sum_{\nu \mid u} \left(\frac{-\Delta}{\nu}\right)}{\Delta^{1/2} \prod_{p \mid \tau,\, p>2} \left(1 - \left(\frac{-\omega}{p}\right) \frac{1}{p} L(1, -\omega)\right)}$$

where

$\chi_2 = 2^{\frac{1}{2}\alpha + 2}$, if $2 \mid \text{æ}$, $0 \leq \alpha \leq \text{æ} - 3$, $2 \mid \alpha$, $m \equiv a \pmod 8$;

$= 0$, if $2 \mid \text{æ}$, $0 \leq \alpha \leq \text{æ} - 3$, $2 \mid \alpha$, $m \not\equiv a \pmod 8$,

or $0 \leq \alpha \leq \text{æ} - 1$, $2 \nmid \alpha$;

$= \left(1 + (-1)^{\frac{m-a}{2}}\right) 2^{\frac{1}{2}\text{æ}}$, if $2 \mid \text{æ}$, $\alpha = \text{æ} - 2$;

$= \left(1 + (-1)^{\frac{1}{2}(m-a)}\right) 2^{\frac{1}{2}\text{æ}}$, if $2 \mid \text{æ}$, $\alpha \geq \text{æ}$, $2 \mid \alpha$, $\Delta_1 \equiv 1 \pmod 4$;

$= 2^{\frac{1}{2}\text{æ}}$, if $2 \mid \text{æ}$, $\alpha = \text{æ}$, $\Delta_1 \equiv -1 \pmod 4$;

$= \left(2 - (-1)^{\frac{1}{4}(\Delta_1 + 1)}\right) 2^{\frac{1}{2}\text{æ}} + \left(1 + (-1)^{\frac{1}{4}(\Delta_1 + 1)}\right)(\alpha - \text{æ} - 2) 2^{\frac{1}{2}\text{æ} - 1}$,

if $2 \mid \text{æ}$, $\alpha > \text{æ}$, $2 \mid \alpha$, $\Delta_1 \equiv -1 \pmod 4$;

$= \left(1 + (-1)^{\frac{1}{4}(\Delta_1 - 1) + \frac{1}{2}(m-a)}\right) 2^{\frac{1}{2}\text{æ}}$, if $2 \mid \text{æ}$, $\alpha \geq \text{æ} + 1$, $2 \nmid \alpha$, $\Delta_1 \equiv 1 \pmod 4$;

$$=\left(1+(-1)^{\frac{1}{4}(\Delta_1+1)}\right)(\alpha-æ-1)2^{\frac{1}{2}æ-1}, \text{ if } 2|æ,$$

$$\alpha\ge æ+1,\ 2\nmid\alpha,\ \Delta_1\equiv -1 (\mathrm{mod}\, 4).$$

$=2^{\frac{1}{2}\alpha+2}$, if $2\nmid æ$, $0\le\alpha\le æ-3$, $2|\alpha$, $m\equiv a (\mathrm{mod}\, 8)$;

$=0$, if $2\nmid æ$, $0\le\alpha\le æ-3$, $2|\alpha$, $m\not\equiv a (\mathrm{mod}\, 8)$,

or $0\le\alpha\le æ-2$, $2\nmid\alpha$;

$$=\left(1+(-1)^{\frac{1}{4}(m-a)}\right)2^{\frac{1}{2}(æ-1)}, \text{ if } 2\nmid æ,\ \alpha\ge æ-1,$$

$$2|\alpha,\ m\equiv\alpha (\mathrm{mod}\, 8);$$

$$\chi_2=\left(1+(-1)^{\frac{1}{4}(m+a)+\frac{1}{2}(m-\Delta_1 a)}\right)2^{\frac{1}{2}(æ-1)}, \text{ if } 2\nmid æ,$$

$$\alpha\ge æ-1,\ 2|\alpha,\ m\equiv -\alpha (\mathrm{mod}\, 4);$$

$$=\left(1+(-1)^{\frac{1}{4}(m-\Delta_1\alpha)}\right)2^{\frac{1}{2}(æ-1)}, \text{ if } 2\nmid æ,\ \alpha\ge æ,$$

$2\nmid\alpha$,

$$m\equiv\Delta_1 a (\mathrm{mod}\, 4);$$

$$=\left(1+(-1)^{\frac{1}{4}(m+\Delta_1 a)+\frac{1}{2}(m-a)}\right)2^{\frac{1}{2}(æ-1)}, \text{ if } 2\nmid æ,$$

$\alpha\ge æ$,

$$2\nmid\alpha,\ m\equiv -\Delta_1 a (\mathrm{mod}\, 4).$$

$$\chi_p=\left(1+\left(\frac{p^{-\beta}na}{p}\right)\right)p^{\frac{1}{2}\beta}, \text{ if } l\ge\beta+1, 2\,|\,\beta;$$

$$=0, \quad \text{if } l\ge\beta+1,\ 2\nmid\beta;$$

$$=\left(1-\left(\frac{-p^{-l}\Delta}{p}\right)\frac{1}{p}\right)\left\{1+\left(1+\left(\frac{-p^{-l}\Delta}{p}\right)\right)\frac{\beta-l}{2}\right\}p^{l/2},$$

$$\text{if } l\le\beta,\ 2\,|\,l,\ 2\,|\,\beta;$$

$$=\left(1-\left(\frac{-p^{-l}\Delta}{p}\right)\frac{1}{p}\right)\left(1+\left(\frac{-p^{-l}\Delta}{p}\right)\right)\frac{\beta-l+1}{2}p^{l/2},$$

$$\text{if } l\le\beta,\ 2\,|\,l,\ 2\nmid\beta;$$

$$=\left\{1+\left(\frac{p^{-l}\Delta}{p}\right)^{\beta+1}\left(\frac{p^{-(\beta+l)}na\Delta}{p}\right)\right\}p^{\frac{1}{2}(l-1)},$$

$$\text{if } l\le\beta,\ 2\nmid l.$$

III. Basic results

The set of forms with discriminant –80 splits into to genera of forms and each genus consists two reduced classes of forms which are respectively

$$f_1=3x_1^2+2x_1x_2+7x_2^2,\quad f_2=3x_1^2-2x_1x_2+7x_2^2$$

and

$$f_3=x_1^2+20x_2^2,\quad f_4=4x_1^2+5x_2^2.$$

P. Kaplan and K.S. Wiliams [6] have obtained formulae for the number of representations of an even positive integer *n* by the forms f_3 and f_4.

It is obvious that

$$r(n;f_1)=r(n;f_2).$$

Thus by Siegel's theorem

$$r(n;f_1)=r(n;f_2)=\frac{1}{2}\rho(n;f_1).$$

The function $\rho(n;f_1)$ may be calculated by the Lemma 2. So we have

Theorem 1. *Let* $n=2^\alpha 5^\beta u$, $(u,10)=1$. *Then*

$$r(n;f_1)=r(n;f_2)=\frac{1}{2}\left(1-\left(\frac{u}{5}\right)\right)\sum_{\nu|u}\left(\frac{-5}{\nu}\right)$$

for $\alpha=0$, $u\equiv 3(\mathrm{mod}\, 4)$,

$$=\left(1-(-1)^\alpha\left(\frac{u}{5}\right)\right)\sum_{\nu|u}\left(\frac{-5}{\nu}\right)$$

for $2|\alpha$, $\alpha\ge 2$, $u\equiv 3(\mathrm{mod}\, 4)$,
or
for $\alpha\ge 2$, $2\nmid\alpha$,
$u\equiv 1(\mathrm{mod}\, 4)$,

$=0$ *otherwise.*

The set of forms with discriminant –128 splits into two genera of forms and each genus consists two reduced classes of forms which are respectively

$$f_5=3x_1^2+2x_1x_2+11x_2^2,\quad f_6=3x_1^2-2x_1x_2+11x_2^2$$

and

$$f_7=x_1^2+32x_2^2,\quad f_8=4x_1^2+4x_1x_2+9x_2^2.$$

P. Kaplan and K.S. Williams [8] have obtained formulae for the number of representations of an even positive integer *n* by the forms f_6 and f_7.

It is obvious that

$$r(n;f_5)=r(n;f_6).$$

Thus by Siegel's theorem

$$r(n;f_5)=r(n;f_6)=\frac{1}{2}\rho(n;f_5).$$

The function $\rho(n;f_5)$ can be calculated by the Lemma 2.

So we have

Theorem 2. *Let* $n=2^\alpha u$, $(u,2)=1$. *Then*

$$r(n;f_5)=r(n;f_6)=\sum_{\nu|u}\left(\frac{-2}{\nu}\right) \text{ for } \alpha=0,$$

$u\equiv 3(\mathrm{mod}\, 8)$,

$$=2\sum_{\nu|u}\left(\frac{-2}{\nu}\right) \quad \text{for } \alpha=2,\ u\equiv 3(\mathrm{mod}\, 8),$$

and

for $\alpha\ge 4$, $u\equiv 1(\mathrm{mod}\, 8)$,
or $u\equiv 3(\mathrm{mod}\, 8)$,

$=0$ *otherwise.*

References

[1] C.L. Siegel, Uber die analytische theorie der quadratischen formen. Annals of mathematics, 36, 1935, 527-606.
[2] K.G. Ramanathan, On the analytic theory of quadratic forms, Acta Arithmetica, 21, 1972, 423-436.
[3] T.V. Vepkhvadze, On a formula of Siegel, Acta Arithmetica, 40, 1981/82, 125-142 (in Russian).
[4] T.V.Vepkhvadze, The representation of numbers by positive Gaussian binary quadratic forms. Trudy Razmadze Tbilissi Mathematical Institute 40, 1971, 21-58 (in Russian).
[5] T. Vepkhvadze, Modular properties of theta-functions and representation of numbers by positive quadratic forms. Georgian Mathematical Journal, 4, 1997, 385-400.
[6] F. Van der Blij, Binary quadratic forms of discriminant – 23, Indagationes Mathematicae, 14, 1952, 489-503.
[7] G.A. Lomadze, On the representation of numbers by binary diagonal quadratic forms, Mathematical sbornik, 68, 119, 1965, 283-312 (in Russian).
[8] P. Kaplan and K.S. Williams, On the number of representations of a positive integer by a binary quadratic form. Acta Arithmetica, 114, 1, 2004, 87-98.